Int J Theor Phys (2010) 49: 564–574
DOI 10.1007/s10773-009-0237-1

The Quantal Canonical Symmetries for a Singular Lagrangian System with Subsidiary Constraints

Ziping Li · Jinhuan Jiang

Received: 29 October 2009 / Accepted: 22 December 2009 / Published online: 9 January 2010

Abstract The quantization for a system containing subsidiary constraints (in configuration space) with a singular Lagrangian is studied, in certain case which can be brought into the theoretical framework of constrained Hamiltonian system. A modified Dirac-Bergmann algorithm for the calculation of all phase-space constraints in those systems is derived. The path integral quantization is formulated by using the Faddeev-Senjanovic scheme. The classical and quantum canonical symmetries (Noether theorem in canonical formalism) are established for such a system. An example is given to illustrate that the connection between the symmetry and conservation law in classical theory are not always validity in the quantum theory.

Keywords Constrained system · Quantum theory · Symmetry and conservation law

1 Introduction

Symmetry is now a fundamental concept in modern physics, classical Noether theorem and their generalization are usually formulated in terms of Lagrange variables in configuration space. Recently the canonical symmetry of a system with a singular Lagrangian both in classical and quantum theory has been presented [1]. A system with a singular Lagrangian is subject to some inherent phase-space constraints in Dirac sense, which is a constrained Hamiltonian (canonical) system. In order to study the quantal canonical symmetry of a constrained Hamiltonian system, the path integral provide a useful tool. The path integral quantization for constrained Hamiltonian system can be formulated with aid of Dirac theory of singular Lagrangian and method of path integral, numerous works were performed, such as Faddeev-Senjanovic scheme [2, 3], Batalin-Fradkin-Vilkovsky method [4] and others [1]. The phase-space path integrals are more fundamental than configuration-space path integrals. In certain integrable cases for canonical momenta, the phase-space path integral can be simplified by carrying out explicit integration over canonical momenta, which can be

Z. Li · J. Jiang (✉)
College of Applied Science, Beijing University of Technology, Beijing 100022, China
e-mail: jiangjh@bjut.edu.cn

converted into configuration-space path integral obtained by using Faddeev-Popov trick [5] (for example, Yang-Mills theory).

A system with a singular Lagrangian is a constrained Hamiltonian system which Lagrangian in configuration space usually does not contain subsidiary constraints. However, there are many physical systems whose motion is subject to some subsidiary constraints in configuration space, such as holonomic/nonhlonomic constraints in mechanical systems, and some subsidiary conditions in field theories (for example, non-linear σ-model, CP^1 model, etc.). We shall investigate here a more general case to study the quantum theory and their quantal canonical symmetries for the systems with the singular Lagrangian containing subsidiary constrained conditions (in configuration space). Those systems are called constrained singular systems.

In order to study the quantum symmetry of the constrained singular systems, first of all one needs to study the quantization of such a system. The constrained singular systems both have inherent phase-space constraints and subsidiary configuration-space constraints. We shall study the following cases: it is supposed that first the subsidiary constraints can be converted into the phase-space constraints by using the definition of canonical momenta, and second, the compatibility between the inherent phase-space constraints and giving subsidiary constraints (expressing in canonical variables) has been taken into account. Thus the theory for certain constrained singular system described in phase space can be brought into the theoretical framework of constrained Hamiltonian system.

The paper is organized as follows. In Sect. 2, the modified Dirac-Bergmann algorithm for calculation of all canonical constraints in a system with a singular Lagrangian containing subsidiary configuration-space constraints is derived. In Sect. 3, the phase-space path integral quantization for constrained singular system is formulated. In Sect. 4, the classical and quantal canonical symmetries for constrained singular systems are studied, the comparison of both results is discussed. In Sect. 5, an example is given to illustrate that the connections between the symmetry and conservation law in classical theories are not always validity in the quantum theories, the conservation law at the quantum level differs from classical one. Sect. 6 is devoted to the conclusions and discussions.

2 The Modified Dirac-Bergmann Algorithm

For the sake of simplicity, let us now first consider the discrete system with finite degree of freedom, the generalization to the field theory is straightforward. Consider a dynamical system described by a singular Lagrangian $L(q^i,\dot{q}^i)$ $(i=1,2,\dots,n)$, we introduce the canonical momenta $p_i=\partial L/\partial\dot{q}^i$ and canonical Hamiltonian $H_c=p_i\dot{q}^i-L$, which may be formed by eliminating only $\dot{q}^i$ (the summation is taken over repeated indices), thus, one can go over from the Lagrange description to the Hamilton description. Consider a system with a singular Lagrangian whose Hessian matrix $[\partial^2L/\partial\dot{q}^i\partial\dot{q}^j]$ is degenerate and suppose its rank to be $n-R$. Then, from the former $n-R$ defining equations of canonical momenta, one can solve generalized velocities $\dot{q}^\sigma$ $(\sigma=1,2,\dots,n-R)$ as the functions of $q^i,\dot{q}^\rho$ and p_a,

$$\dot{q}^\sigma=f^\sigma(q^i,\dot{q}^\rho,p_a)\quad(\sigma,a=1,2,\dots,n-R,\rho=1,2,\dots,R)\tag{1}$$

substituting the $\dot{q}^\sigma$ in the last R defining equations for the canonical momenta yields R relations (primary constraints) between the canonical variables [6]

$$\phi_a^0(q^i,p_i)=0\quad(a=1,2,\dots,R)\tag{2}$$

It is supposed that for such a system whose motion is also subject to some subsidiary configuration-space constraints

$$G_s(q^i, \dot{q}^i) = 0 \quad (s = 1, 2, \ldots, m', a + m' < n) \tag{3}$$

the equations of motion in configuration space of this system are given by [7]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \lambda^s \frac{\partial G_s}{\partial \dot{q}^i} \tag{4}$$

where $\lambda^s(t)$ are Lagrange multipliers.

Supposed that those systems are described by singular Lagrangians, and moreover, are submitted to some subsidiary constraints (3), now we study those cases that one substitutes (1) into (3), which can be converted to the canonical constraints $G_s(q^i, p_i) = 0$. Let us suppose that we have taken into account the compatibility between the inherent constraints arising from the singularity of the Lagrangian and the giving subsidiary constraints (3) in canonical variables. From the variation of the canonical Hamiltonian, one gets

$$\delta H_c = \dot{q}^i \delta p_i - \frac{\partial L}{\partial q^i}\delta q^i \tag{5}$$

The canonical Hamiltonian is only a function of canonical variables q^i and p_i both for regular and singular Lagrangian [8], thus, the variation of canonical Hamiltonian H_C can be written as

$$\delta H_c = \frac{\partial H_C}{\partial q^i}\delta q^i + \frac{\partial H_C}{\partial p_i}\delta p_i \tag{6}$$

Comparing (5) and (6), one obtains

$$\left(\dot{q}^i - \frac{\partial H_C}{\partial p_i}\right)\delta p_i - \left(\frac{\partial L}{\partial q^i} + \frac{\partial H_C}{\partial q^i}\right)\delta q^i = 0 \tag{7}$$

Using (4), the expression (7) can be written as

$$\left(\dot{q}^i - \frac{\partial H_C}{\partial p_i}\right)\delta p_i - \left(\dot{p}_i + \frac{\partial H_C}{\partial q^i} - \lambda^s \frac{\partial G_s}{\partial \dot{q}^i}\right)\delta q^i = 0 \tag{8}$$

In the mechanical systems, usually the Lagrangians of the systems are quadratic form in $\dot{q}^i$, thus $\partial p_j/\partial \dot{q}^i$ are independent of $\dot{q}^i$, and $\frac{\partial G_s}{\partial \dot{q}^i} = \frac{\partial G_s}{\partial p_j}\frac{\partial p_j}{\partial \dot{q}^i} \equiv G_{si}(q^i, p_i)$ are functions of canonical variables if one adopts above hypothesis. For the systems with singular Lagrangian, from (2) it follows that

$$\frac{\partial \phi_a^0}{\partial q^i}\delta q^i + \frac{\partial \phi_a^0}{\partial p_i}\delta p_i = 0 \tag{9}$$

Introducing the Lagrange multipliers $\mu^a(t)$ and combining (8) and (9), one obtains the equations of motion in phase space for a system with a singular Lagrangian containing subsidiary constraints(which can be converted into canonical constraints by using the definition of canonical momenta)

$$\dot{q}^i = \frac{\partial H_C}{\partial p_i} + \mu^a \frac{\partial H_C}{\partial p_i} \tag{10a}$$

$$\dot{p}_i = -\frac{\partial H_C}{\partial q^i} - \mu^a \frac{\partial H_C}{\partial p_i} + \lambda^s G_{si} \tag{10b}$$

For any function $F(q^i, p_i)$ of canonical variables q^i and p_i, using (10), one can obtain

$$\dot{F} = \frac{\partial F}{\partial q^i}\dot{q}^i + \frac{\partial F}{\partial p_i}\dot{p}_i = \{F, H_T\} + G_i \frac{\partial F}{\partial p_i} \tag{11a}$$

Where $H_T = H_C + \mu^a \phi_a^0$ is a total Hamiltonian, $G_i = \lambda^s G_{si}$, and $\{.,.\}$ denotes Poisson bracket.

Let it be considered that $\phi_a^0 = \phi_a^0(q^i, p_i)$ and $G_s(q^i, p_i) = 0$ are primary constraints for those constrained singular systems [9] and denoted by $\Phi_\alpha^0 \equiv (\phi_a^0, G_s)$, the consistency conditions of primary constraints, $\dot{\Phi}_\alpha^0 = 0$, may give some equations to determine the Lagrange multipliers $\mu^a(t)$ or $\lambda^s(t)$, or perhaps, may yields secondary constraints

$$\Phi_\alpha^1 = \dot{\Phi}_\alpha^0 = \{\Phi_\alpha^0, H_T\} + G_i \frac{\partial \Phi_\alpha^0}{\partial p_i} = 0 \tag{12}$$

The consistency conditions of secondary constraints can also lead to some new secondary constraints, this procedure enables us to define successively the secondary constraints

$$\Phi_\alpha^k = \dot{\Phi}_\alpha^{k-1} = \{\Phi_\alpha^{k-1}, H_T\} + G_i \frac{\partial \Phi_\alpha^{k-1}}{\partial p_i} = 0 \tag{13}$$

until

$$\Phi_\alpha^{m+1} = \dot{\Phi}_\alpha^m = \{\Phi_\alpha^m, H_T\} + G_i \frac{\partial \Phi_\alpha^m}{\partial p_i} = C_{\alpha k}^\beta \Phi_\beta^k \tag{14}$$

are satisfied after a finite number steps. This is a modified Dirac-Bergmann algorithm for the calculation of the phase-space constraints in a system containing subsidiary constraints with a singular Lagrangian.

It is also supposed that we have taken into account the compatibility among all constraints $\{\Phi_\alpha^k\}$ $(\alpha = a, s, k = 0, 1, 2, \dots, m)$.

For a system with a singular Lagrangian containing subsidiary holonomic constraints $G_s(q^i) = 0$, in this case the equations of motion in configuration space can be obtained by using $\lambda^s \frac{\partial G_s}{\partial q^i}$ instead of $\lambda^s \frac{\partial G_s}{\partial \dot{q}^i}$ in (4), and the canonical equations can be written as $\dot{q}^i = \{q^i, H_T'\}$, $\dot{p}_i = \{p_i, H_T'\}$, where $H_T' = H_C + \mu^a \phi_a^0 + \lambda^s G_s$. A slight modified Dirac-Bergmann algorithm for the calculation all phase-space constraints in this holonomic system is very similar as usual singular Lagrangian without containing subsidiary constraint, which can be obtained by using H_T' instead of H_T in usual one.

Let us consider a system with a singular Lagrangian containing the non-holonomic constraints linear in $\dot{q}^i$, $G_s = \dot{q}^i \frac{\partial \bar{G}_s}{\partial q^i} + b_s$, where $\bar{G}_s$ and b_s are functions of q^i, only, in this case, (11a) can be written as

$$\dot{F} = \{F, H_T\} + \lambda_s \frac{\partial \bar{G}_s}{\partial q^i}\frac{\partial F}{\partial p_i} = \{F, H_C + \mu^a \phi_a^0 + \lambda_s \bar{G}_s\} \tag{11b}$$

Thus, this non-holonomic system can be brought into theoretical framework of constrained Hamiltonian system if those constraints $G_s(q^i, \dot{q}^i) = 0$ can be converted into the canonical constraints by using the definition of canonical momenta and the compatibility among all constraints $\{\Phi_\alpha^k\}$ have been taken into account. A slight modified Dirac-Bergman algorithm for the calculation all phase-space constraints in this non-holonomic system can be obtained by using $H''_T = H_C + \mu^a \phi^0_a + \lambda_s \bar{G}_s$ instead of total Hamiltonian H_T in usual one.

All the constraints $\{\Phi^k_\alpha\}$ are classified into two classes, a Φ^k_α is defined to be first class if $\{\Phi^k_\alpha, \Phi^{k'}_\beta\} = 0(\mathrm{mod}\,\Phi^l_r)$ for all Φ^l_r, otherwise it is second class, first-class constraints are linked to gauge freedom as usual [1].

3 The Quantization for Constrained Singular Systems

The path-integral formulation is a useful tool for investigating the quantal symmetries of the system. In order to study the quantal symmetry for a constrained singular system, the first step is to give its quantization. One method of the quantization of the system is the canonical-operator quantization, another one known as the Feynman path integral and its generalization. The advantage of the path integral formulation to study the symmetry properties of the system at the quantum level is that one deals C-number, not q-number. Phase-space path integrals are more fundamental than configuration-space one. In the phase-space integral approach, one starts with a definite Hamiltonian to provide a dynamical framework. Nevertheless, in contrast with conventional approach, the equation of motion of the system will not enter the calculation; the final results satisfy automatically restrictions of the equations of motion.

The quantization of the constrained Hamiltonian system based on Dirac formalism and using the method of path integral was performed by Faddeev [2], he restricted the discussion to the case when only first-class constraints are present. A generalization of Faddeev's method to a system, which also contains second-class constraints, was done by Senjanovic [3]. For the case when a constrained singular system which can be described in canonical variables and can be brought into the theoretical framework of constrained Hamiltonian system, the Faddeev-Senjanovic quantization scheme can be applied to those constrained singular system. Let $\{\Phi^k_\alpha\}$ $(\alpha = a, s, k = 0, 1, 2, \ldots m)$ denotes all canonical constraints derived by using a modified Dirac-Bergmann algorithm for a constrained singular system, we separate all constraints $\{\Phi^k_\alpha\}$ into first-class constraints and second ones, let $\Lambda_l(q^i, p_i) = 0$ $(l = 1, 2, \ldots, k_1)$ be first-class constraints which linking to gauge freedom is assumed, and $\theta_i(q^i, p_i) = 0$ $(i = 1, 2, \ldots, I_1)$ be second-class constraints. According to Faddeev-Senjanovic path integral quantization scheme, for each first-class constraints, one must choose a gauge condition, the gauge conditions connecting with the first-class constraints are denoted by $\Omega_k(q^i, p_i) = 0$ $(k = 1, 2, \ldots, k_1)$. Then the quantum transition amplitude (S-matrix element) for those systems is given by [3]

$$Z[0] = \int \mathcal{D}q^i \mathcal{D}p_i \prod_{k=1}^{k_1} \delta(\Lambda_k)\delta(\Omega_k) \cdot \prod_{i=1}^{2I_1} \delta(\theta_i) \cdot \det|\{\Lambda_l, \Omega_k\}| \cdot [\det|\{\theta_i, \theta_j\}|]^{1/2} \times \exp\left\{i \int dt[p_i \dot{q}^i - H_c]\right\} \tag{15}$$

where H_C is a canonical Hamiltonian.

The quantum field theory can be considered to be characterized by the set of all the Green's functions which can be written compactly using the path integral. This technique consists of canonical Lagrangian $L^P = p_i \dot{q}^i - H_C$ by addition of the terms $q^i J_i$, where $J_i(t)$ are exterior source for the $q^i(t)$. Thus, the phase-space generating functional of the Green's functions can be written as

$$Z[J]=\int \mathcal{D}q^i\mathcal{D}p_i\prod_{j,k,l}\delta(\theta_j)\delta(\Lambda_k)\delta(\Omega_l)\cdot\det|\{\Lambda_k,\Omega_l\}|\cdot[\det|\{\theta_i,\theta_j\}|]^{1/2}$$
$$\times\exp\left\{i\int dt[p_i\dot{q}^i-H_c+J_iq^i]\right\} \tag{16}$$

This path integral main ingredient is the classical action and exterior sources together with the measure in the phase space of field configurations. From this generating functional one can derive all quantities, such as quantal conserved quantities in the symmetries of the system, etc. Using the δ-functions and integral properties of Grassmann variables $\eta^+(t)$ and $\eta(t)$, the expression (16) can be written as [1]

$$Z[J]=\int \mathcal{D}q^i\mathcal{D}p_i\mathcal{D}\eta^+\mathcal{D}\eta\mathcal{D}\lambda_m\cdot\exp\left\{i\int dt[L^P_{eff}+J_iq^i]\right\} \tag{17}$$

where

$$L^P_{eff}=L^P+L_m+L_{gh} \tag{18a}$$

$$L^P=p_i\dot{q}^i-H_c \tag{18b}$$

$$L_m=\lambda_j\theta_j+\lambda_k\Lambda_k+\lambda_l\Omega_l,\qquad \lambda_m=(\lambda_j,\lambda_k,\lambda_l) \tag{18c}$$

$$L_{gh}=\int d\tau\left[\eta^+(t)\{\Lambda_k(t),\Omega_l(\tau)\}\eta_l(\tau)+\frac{1}{2}\eta^+_i(t)\{\theta_i(t),\theta_j(\tau)\}\eta_j(\tau)\right] \tag{18d}$$

4 The Canonical Symmetries

The classical/quantal canonical symmetries for a system with a singular Lagrangian not containing subsidiary constraints were studied in previous works [1, 10]. Now we give a generalization of those discussions to a system with a singular Lagrangian containing subsidiary conditions, the comparisons of the results of both classical and quantal theory are investigated.

Let us consider a global infinitesimal transformation

$$\begin{cases} t'=t+\Delta t=t+\varepsilon_\sigma\tau^\sigma(t,q^i,p_i)\\ q^{i'}(t')=q^i(t)+\Delta q^i(t)=q^i(t)+\varepsilon_\sigma\xi^{i\sigma}(t,q^i,p_i)\\ p'_i(t')=p_i(t)+\Delta p_i(t)=p_i(t)+\varepsilon_\sigma\eta^\sigma_i(t,q^i,p_i)\end{cases} \tag{19}$$

where ε_σ $(\sigma=1,2,\ldots,r)$ are infinitesimal constant parameters, and $\tau^\sigma,\xi^{i\sigma},\eta^\sigma_i$ are some functions of canonical variables. It is supposed that the canonical Lagrangian L^P is invariant under the transformation (19), under this transformation one has [10]

$$\left(\dot{q}^i-\frac{\partial H_C}{\partial p_i}\right)\delta p_i-\left(\dot{p}_i+\frac{\partial H_C}{\partial q^i}\right)\delta q^i+\frac{d}{dt}[p_i(\Delta q^i-\dot{q}^i\Delta t)+(p_i\dot{q}^i-H_C)\Delta t]=0 \tag{20}$$

Let it be further assumed that the simultaneous $\delta q^i=\Delta q^i-\dot{q}^i\Delta t$ and $\delta p_i=\Delta p_i-\dot{p}_i\Delta t$ determined by the transformation (19) satisfy (9), and δq^i satisfy the same conditions as virtual displacement imposed by subsidiary constraints (3)

$$\frac{\partial G_s}{\partial \dot{q}^i}\delta q^i = 0 \tag{21}$$

Using a set of Lagrange multipliers $\mu^a(t)$ and $\lambda^s(t)$, combining the expressions (9), (20) and (21), one gets

$$\begin{aligned}&\left(\dot{q}^i - \frac{\partial H_C}{\partial p_i} - \mu^a \frac{\partial \phi_a^0}{\partial p_i}\right)\delta p_i - \left(\dot{p}_i + \frac{\partial H_C}{\partial q^i} + \mu^a \frac{\partial \phi_a^0}{\partial q^i} - \lambda^s \frac{\partial G_s}{\partial \dot{q}^i}\right)\delta q^i \\ &\quad + \frac{d}{dt}[p_i \Delta q^i - H_C \Delta t] = 0\end{aligned} \tag{22}$$

Using the equations of motion (10), from (22), one has

$$p_i \xi^{i\sigma} - H_C \tau^\sigma = \text{const} \tag{23}$$

Thus, we have the following classical canonical Noether theorem for a system with a singular Lagrangian containing subsidiary constraints (3): If, under the transformation (19) the canonical Lagrangian L^P is invariant, and the simultaneous variations of canonical variables δq^i and δp_i determined by the transformation (19) satisfy (9) and (21), then, there are some conservation laws (23) for those systems.

Based on the generating functional of the Green's function (17), let us now study the quantum canonical symmetry of the constrained singular system. It is supposed that the effective canonical action determined by (18a) is invariant under the global infinitesimal transformation (19). Now we localize the transformation (19) which can be done by using $\varepsilon_\sigma(t)$ instead of ε_σ in the transformation (19), and consider the following local infinitesimal transformation connected with the transformation (19)

$$\begin{cases} t' = t + \Delta t = t + \varepsilon_\sigma(t)\tau^\sigma(t, q^i, p_i) \\ q^{i'}(t') = q^i(t) + \Delta q^i(t) = q^i(t) + \varepsilon_\sigma(t)\xi^{i\sigma}(t, q^i, p_i) \\ p_i'(t') = p_i(t) + \Delta p_i(t) = p_i(t) + \varepsilon_\sigma(t)\eta_i^\sigma(t, q^i, p_i) \end{cases} \tag{24}$$

where $\varepsilon_\sigma(t)$ $(\sigma = 1, 2, \ldots, r)$ are infinitesimal arbitrary functions and their values and derivatives vanish on the end point of the time interval. According to assumption the effective canonical action I^P_{eff} is invariant under the transformation (19) and using the boundary conditions of the $\varepsilon_\sigma(x)$, one can obtain the variation of the effective canonical action I^P_{eff} under the transformation (24) [1]

$$\begin{aligned}\Delta I^P_{eff} &= \int \left\{\frac{\delta I^P_{eff}}{\delta q^i}\delta q^i + \frac{\delta I^P_{eff}}{\delta p_i}\delta p_i + \frac{d}{dt}[p_i \delta q^i + (p_i \dot{q}^i - H_{eff})\Delta t]\right\} dt \\ &\quad + \int \left\{[p_i(\xi^{i\sigma} - \dot{q}^i \tau^\sigma) + (p_i \dot{q}^i - H_{eff})\tau^\sigma]\frac{d}{dt}\varepsilon_\sigma(t)\right\} dt \\ &= -\int dt \varepsilon_\sigma(t) \frac{d}{dt}[p_i(\xi^{i\sigma} - \dot{q}^i \tau^\sigma) + (p_i \dot{q}^i - H_{eff})\tau^\sigma]\end{aligned} \tag{25}$$

where

$$\frac{\delta I^P_{eff}}{\delta q^i} = -\dot{p}_i - \frac{\partial H_{eff}}{\partial q^i}, \quad \frac{\delta I^P_{eff}}{\delta p_i} = \dot{q}^i - \frac{\partial H_{eff}}{\partial p_i}$$

and H_{eff} is a Hamiltonian connecting with an effective canonical Lagrangian (18a). In the expression (25) we have use that the effective canonical action I^P_{eff} is invariant under the transformation (19), thus, the first integral in (25) is equal to zero, and the integration by part of second integral in the right-hand side of (25) for the first equality, since $\varepsilon_\sigma(t)$ are vanish in the end point, therefore we have the expression (25).

It is supposed that the Jacobian of the local transformation (24) connected with the global transformation (19) is equal to unity, the generating functional (17) is invariant under the transformation (24), thus, one has

$$Z[J] = \int \mathcal{D}q^i \mathcal{D}p_i \mathcal{D}\eta^+ \mathcal{D}\eta \mathcal{D}\lambda_m \cdot \exp\left\{ i \int dt [L^P_{eff} + J_i q^i] \right\} \\ \times \left\{ 1 - i \int dt \varepsilon_\sigma(t) \left[\frac{d}{dt}(p_i \xi^{i\sigma} - H_{eff}\tau^\sigma) + J_i(\xi^{i\sigma} - \dot{q}^i \tau^\sigma) \right] \right\} \tag{26}$$

Functionally differentiating (26) with respect to $\varepsilon_\sigma(t)$, and let $J_i = 0$, one obtains

$$\langle 0 | T^*[p_i \xi^{i\sigma} - H_{eff}\tau^\sigma] | 0 \rangle = \text{const} \quad (\sigma = 1, 2, \ldots, r) \tag{27}$$

where the symbol $|0\rangle$ indicates the ground state of the system, and the symbol T^* stands for the covariant T product [11]. Consequently, if the effective canonical action I^P_{eff} is invariant under the transformation (19) and the Jacobian of corresponding transformation (24) is equal to unity, then, there are conservation laws (27) at the quantum level for a system containing subsidiary constraints with a singular Lagrangian. This conservation laws differ from the classical ones, in the latter case provide a canonical Hamiltonian H_C, whereas the former case provide an effective Hamiltonian H_{eff} which may include all constraints (subsidiary constraints, especially) and gauge conditions arising from the quantization of the constrained singular system.

5 An Example

Let us consider a system whose Lagrangian is given by [12]

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) + \frac{1}{2}q_3^2(q_1^2 + q_2^2) - q_3(q_1\dot{q}_2 - q_2\dot{q}_1) - \frac{1}{2}(q_1^2 + q_2^2) \tag{28}$$

Suppose that this system is subject to a subsidiary constraint

$$G = a^2\dot{q}_1^2 - \dot{q}_1\dot{q}_2 - \dot{q}_2^2 = 0 \tag{29}$$

where a is a constant. The canonical momenta are given by

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = \dot{q}_1 + q_2 q_3, \qquad p_2 = \dot{q}_2 - q_1 q_3, \qquad p_3 = 0 \tag{30}$$

Thus, the Lagrangian (28) is singular and primary constraint is

$$\phi^0 = p_3 = 0 \tag{31}$$

The canonical Hamiltonian is given by

$$H_C = \dot{p}_i \dot{q}^i - L = \frac{1}{2}(p_1^2 + p_2^2) - q_3(p_1 q_2 - p_2 q_1) + \frac{1}{2}(q_1^2 + q_2^2) \tag{32}$$

From (30) one can solve $\dot{q}_1$ and $\dot{q}_2$ as the function of canonical variables and substitute them into the (29), thus, the subsidiary constraint (29) can be converted to canonical constraint in phase space.

$$G^0 = a^2(p_1 - q_2q_3)^2 - (p_1 - q_2q_3)(p_2 + q_1q_3) - (p_2 + q_1q_3)^2 = 0 \tag{33}$$

Obviously, constrains (31) and (33) are compatibility.

The constraint (33) can be considered as a primary constraint in phase space. Using the above modified Dirac-Bergmann algorithm, the consistency conditions of primary constraints $\phi^0 = 0$ and $G^0 = 0$, $\dot{\phi}^0 = 0$ and $\dot{G}^0 = 0$, give some equations to determine the Lagrange multipliers $\mu(t)$ and $\lambda(t)$ respectively, and do not yield any secondary constraint, and

$$\{\phi^0, G^0\} = 2a^2q_2(p_1 - q_2q_3) - q_1p_1 - q_2p_2 - 2q_1q_2q_3 + 2q_1(p_2 + q_1q_3) \tag{34}$$

thus $\phi^0 = 0$ and $G^0 = 0$ are second-class constrains. Using the Faddeev-Senjanovic path-integral quantization scheme, the phase-space generating functional of Green's can be written as

$$\begin{aligned} Z[J] &= \int \mathcal{D}q^i\mathcal{D}p_i\mathcal{D}\eta^+\mathcal{D}\eta\mathcal{D}\lambda_m\delta(\phi^0)\delta(G^0)\sqrt{\det|\{\phi^0, G^0\}|} \\ &\quad \times \exp\left\{i\int dt[p_i\dot{q}^i - H_C + J_iq^i)\right\} \\ &= \int \mathcal{D}q^i\mathcal{D}p_i\mathcal{D}\eta^+\mathcal{D}\eta\mathcal{D}\lambda_m \exp\left\{i\int dt[L^P_{eff} + J_iq^i]\right\} \end{aligned} \tag{35}$$

where

$$L^P_{eff} = L^P + L_m + L_{gh} \tag{36a}$$

$$L^P = p_i\dot{q}^i - H_c \tag{36b}$$

$$L_m = \lambda_1\phi^0 + \lambda_2G^0 \quad (\lambda_1, \lambda_2 \text{ are multipliers}) \tag{36c}$$

$$L_{gh} = \frac{1}{2}\int d\tau\eta_i^+(t)\{\theta_i(t), \theta_j(\tau)\}\eta_j(\tau) \tag{36d}$$

and $\theta_1 = \phi^0$ and $\theta_2 = G^0$. Thus,

$$\det|\{\theta_i, \theta_j\}| = \left(\{\phi^0, G^0\}\right)^2 \tag{37}$$

The Lagrangian (28) and the primary constraint (31) are invariant under the time translation $t' = t - \varepsilon$, because the subsidiary constraint is a homogeneous function of $\dot{q}^i$ in two order, thus

$$\frac{\partial G}{\partial \dot{q}^i}\delta q^i = \varepsilon\frac{\partial G}{\partial \dot{q}^i}\dot{q}^i = 2\varepsilon G = 0 \tag{38}$$

under the time translation. According classical canonical Noether theorem, from (23), one obtains ($\tau^\sigma = 1, \xi^{i\sigma} = 0$) the energy conservation of this constrained singular system at the classical level,

$$H_C = \frac{1}{2}(p_1^2 + p_2^2) - q_3(p_1q_2 - p_2q_1) + \frac{1}{2}(q_1^2 + q_2^2) = \text{const} \tag{39}$$

This result can also be derived by using the integral theory of non-holonomic system in configuration space [13].

Let us now discuss the quantum case. The effective canonical Lagrangian L_{eff}^P is also invariant under the time translation and Jacobian of the corresponding transformation of the canonical variables is equal to unity. From (27) one gets a conservation law

$$\langle 0|T^* H_{eff}|0\rangle = \text{const} \tag{40}$$

for this constrained singular system at the quantum level. The conservation law (40) differs from classical one (39). This means that the connection between symmetry and conservation law in classical theories in general is no longer preserved in quantum theories. This case is called anomaly. It had been pointed out that the anomalies could be viewed as a result of the non-invariance of the functional measure under the some symmetry transformation [14]. The above result indicates that the anomaly may appear in a case with invariance of the functional measure under the symmetry transformation.

6 Conclusions and Discussions

The quantization for a system containing subsidiary constraints with a singular Lagrangian is studied. We discuss in the following cases. (1) The subsidiary constraints can be converted into the canonical constraints by using the definition of canonical momenta. (2) The compatibility between the subsidiary constraints and inherent constraints arising from the singularity of the Lagrangian had been taken into account. In those cases, a modified Dirac-Bergmann algorithm for a constrained singular system is deduced. The path integral quantization is formulated by using Faddeev-Senjanovic scheme. The canonical symmetries are study both at the classical and quantum level and an example is given to show that the classical result is not always validity at the quantum level.

For a system with a regular Lagrangian containing subsidiary constraints, the all $\dot{q}^i$ can be solved from the definition of the canonical momenta, thus, the subsidiary constraints can be inverted into the phase-space constraints. This system can be brought into the theoretical framework of constrained Hamiltonian system.

In the study of the properties of the electromagnetic field near the interface of dielectric media, the boundary conditions $G_s(A^\mu_{,\nu}) = 0$ on the interface have been used as the subsidiary conditions [15, 16]. These subsidiary constraints can be inverted into the canonical constraints by using the definition of canonical momenta. It is easy to check that these subsidiary canonical constraints are compatibility with the inherent phase-space constraints arising from the singularity of Lagrangian of the electromagnetic field. Using the modified Dirac-Bergmann algorithm, the quantization for those constrained singular system can be formulated by using Faddeev-Senjanovic scheme and quantum canonical symmetries for those system can also be studied. Work along this line is in progress.

References

1. Li, Z.P., Jiang, J.H.: Symmetries in Constrained Canonical Systems. Science, Beijing (2002)
2. Faddeev, L.D.: Theor. Math. Phys. **1**, 1 (1970)
3. Senjanovic, P.: Ann. Phys. **100**, 229 (1976)
4. Henneaux, M.: Phys. Rep. **126**, 1 (1985)
5. Faddeev, L.D., Popov, V.N.: Phys. Lett. B **25**, 29 (1967)

6. Castellani, L.: Ann. Phys. **143**, 357 (1983)
7. Mei, F.X.: The Foundation of Non-holonomic System Mechanics. Beijing Institute of Technology Press, Beijing (1985)
8. Li, Z.P.: J. Phys., Math. Gen. A **24**, 4261 (1991)
9. Banerjee, R.: Nucl. Phys. B **419**, 611 (1994)
10. Li, Z.P.: Classical and Quantum Dynamics of Constrained Systems and Their Symmetry Properties. Beijing Polytechnic University Press, Beijing (1993)
11. Suura, H., Young, B.L.: Phys. Rev. D **8**, 875 (1973)
12. Li, Z.P., Wu, B.C.: Int. J. Theor. Phys. **33**, 1063 (1994)
13. Li, Z.P.: Nature **11**, 554 (1988) (in Chinese)
14. Fujikawa, K.: Phys. Rev. Lett. **44**, 1733 (1980)
15. Liu, X.Y.: J. Phys., Math. Gen. A **23**, L323 (1991)
16. Li, Z.P., Wu, B.C.: Commun. Theor. Phys. **24**, 251 (1995)